Review of Accelerator Physics Concepts

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Beam Loss Machine Protection Course

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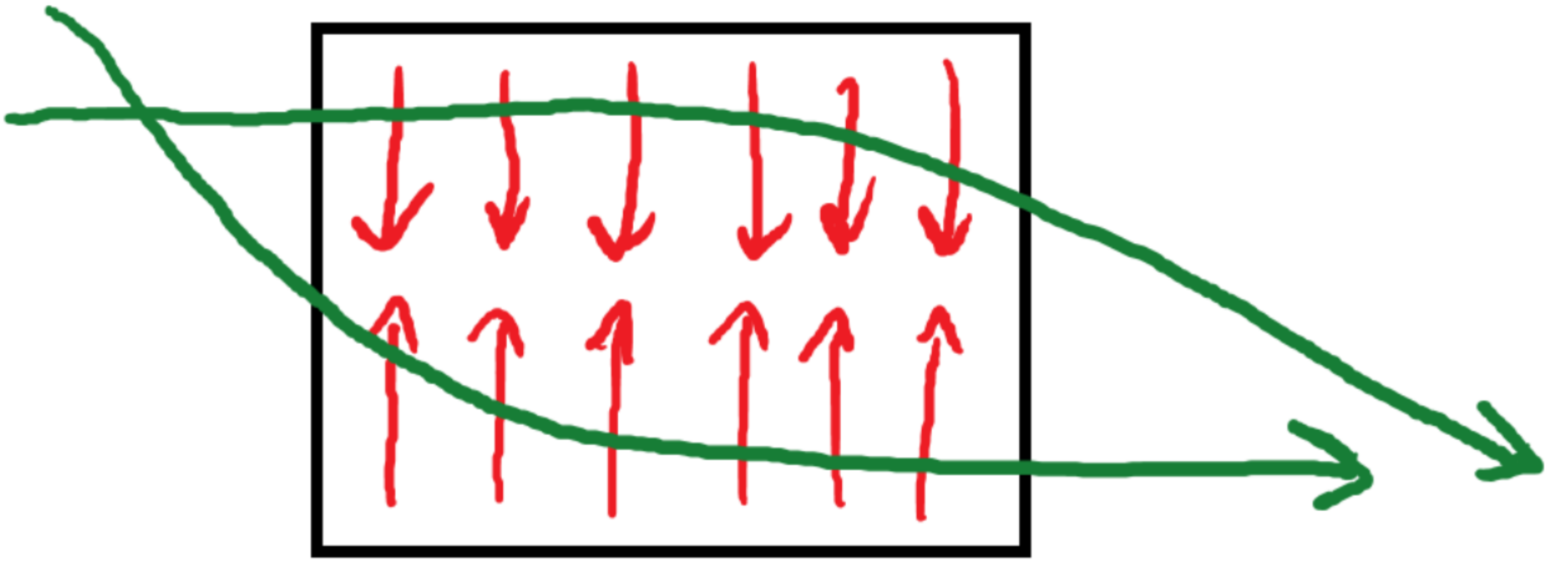
Overview

1. Betatron Motion
2. Nonlinearities
3. Off-Momentum Particles
4. RF Capture

Betatron Motion

Linear Focusing

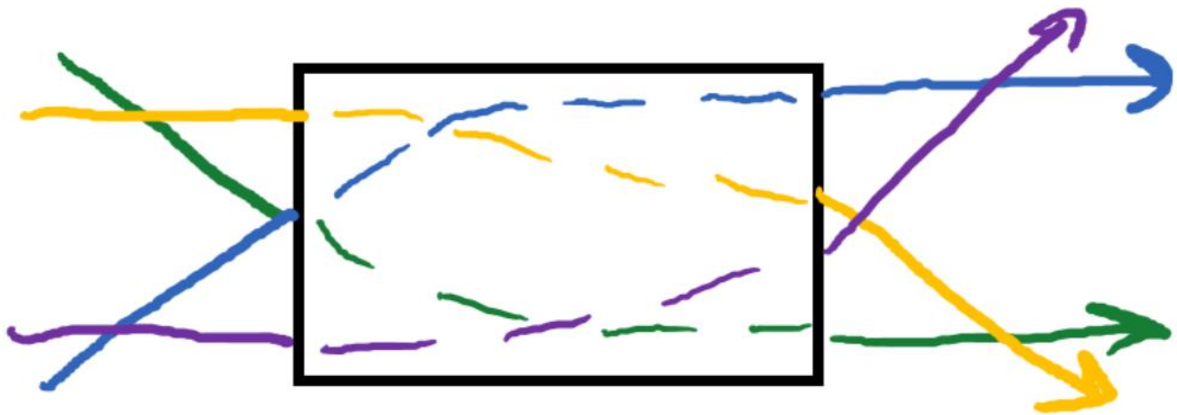
We can solve the linear Hill's equation: $x'' + K(s)x = 0$



$$x(s) = \begin{cases} x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s), & K < 0 \\ x_0 + x_0' s, & K = 0 \\ x_0 \cos(\sqrt{K}s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}s), & K > 0 \end{cases}$$

Transfer Matrices

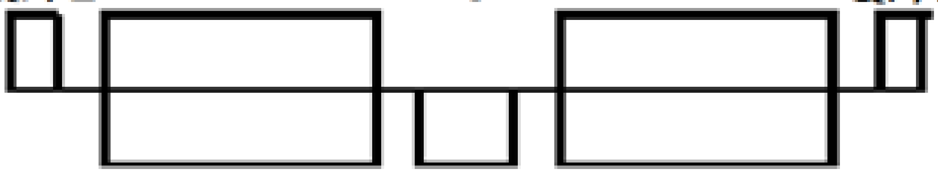
$$x(s) = \begin{cases} x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s), & K < 0 \\ x_0 + x_0' s, & K = 0 \\ x_0 \cos(\sqrt{K}s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}s), & K > 0 \end{cases}$$



The final position and slope is a linear combination of the initial position and initial slope. We can use matrices:

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

Example: FODO Cell



$$
\begin{aligned}
\mathbf{M} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}
\end{aligned}
$$

The transfer matrix for a sequence of elements can be obtained by multiplying the matrices for the components.

TWISS (Courant-Snyder) Parameters

The transfer matrix for a stable ring can be parametrized:

$$M(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\alpha_x = -\frac{\beta_x'}{2}$$

$$\gamma_x = \frac{1+\alpha_x^2}{\beta_x}$$

The transfer matrix for a general transfer matrix:

$$M(s_2|s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\Phi + \alpha_1\sin\Delta\Phi) & \sqrt{\beta_1\beta_2}\sin\Delta\Phi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\Phi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\Phi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\Phi - \alpha_2\sin\Delta\Phi) \end{pmatrix}$$

For example, we can calculate TWISS for a FODO ring:

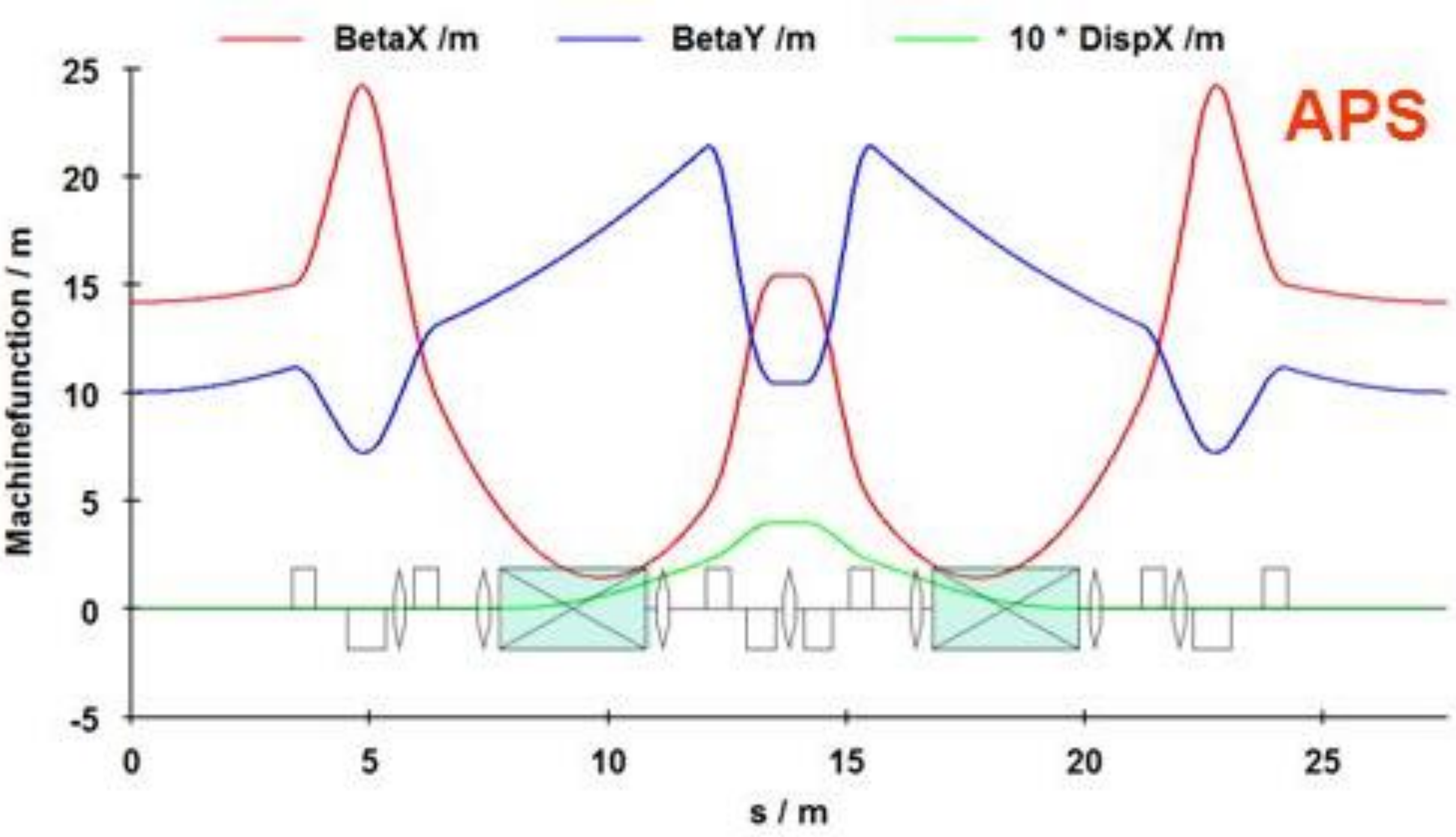
$$M(s_0) = \begin{pmatrix} 1-\frac{L^2}{2f} & 2L\left(1+\frac{L}{2f}\right) \\ -\frac{L}{2f^2}\left(1-\frac{L}{2f}\right) & 1-\frac{L^2}{2f} \end{pmatrix}$$

$$\cos\Phi = \frac{1}{2}\mathrm{Tr}(M) = 1 - L^2/2f$$

$$\beta = \frac{(2L)(1+L/2f)}{\sin\Phi}$$

$$\alpha = 0$$

TWISS Plots


BetaX /m
BetaY /m
10 * DispX /m
APS
Machinefunction / m
s / m
25
20
15
10
5
0
-5
0
5
10
15
20
25

TWISS & Betatron Oscillation

The general transfer matrix can be decomposed:

$$M(s_2|s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\Phi + \alpha_1 \sin\Delta\Phi) & \sqrt{\beta_1\beta_2}\sin\Delta\Phi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\Phi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\Phi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\Phi - \alpha_2\sin\Delta\Phi) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\Delta\Phi & \sin\Delta\Phi \\ -\sin\Delta\Phi & \cos\Delta\Phi \end{pmatrix} \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix}^{-1}$$

An inverse transformation, a rotation, and transformation.

This allows us to understand what the TWISS parameters mean for the particle motion:

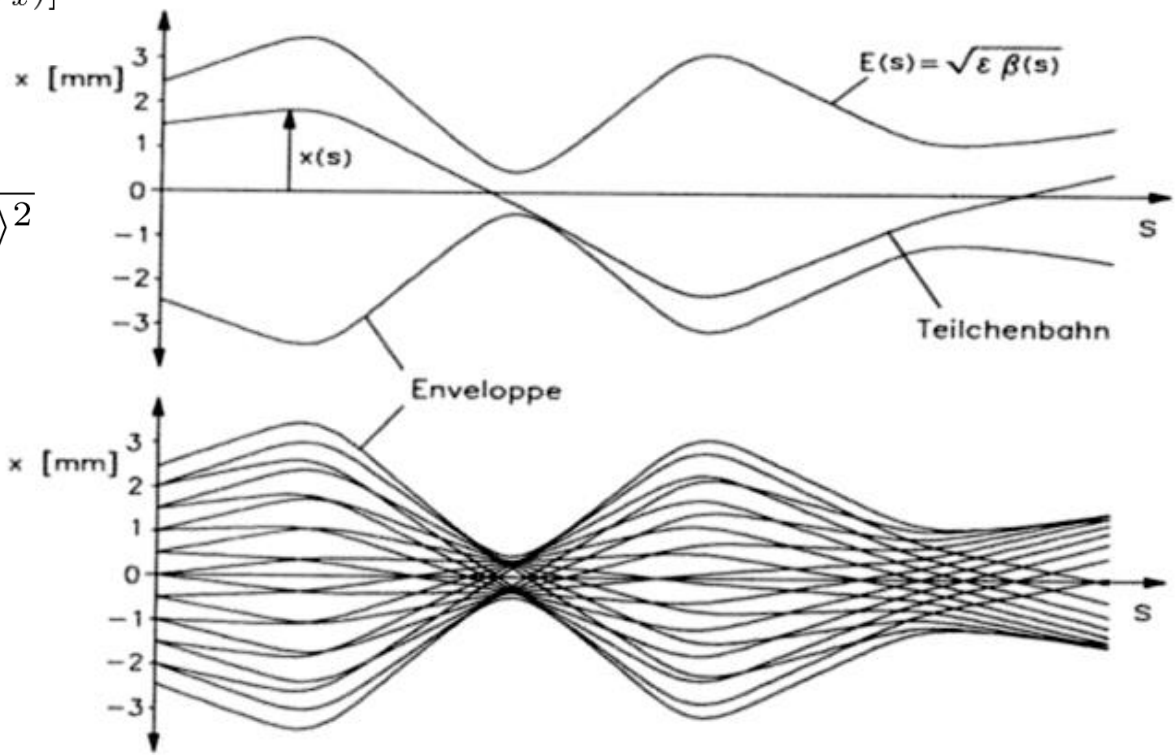
$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x) \qquad x' = -\sqrt{\frac{2J_x}{\beta_x}}[\sin(\phi_x) + \alpha_x\cos(\phi_x)]$$

Betatron Oscillation

$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x)$$

$$\alpha_x = -\frac{\beta'_x}{2} \qquad \gamma_x = \frac{1+\alpha_x^2}{\beta_x}$$

$$x' = -\sqrt{\frac{2J_x}{\beta_x}}[\sin(\phi_x) + \alpha_x\cos(\phi_x)]$$



Emittance:

$$\epsilon_{rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\epsilon_{99\%} = 2.576^2\epsilon_{rms}$$

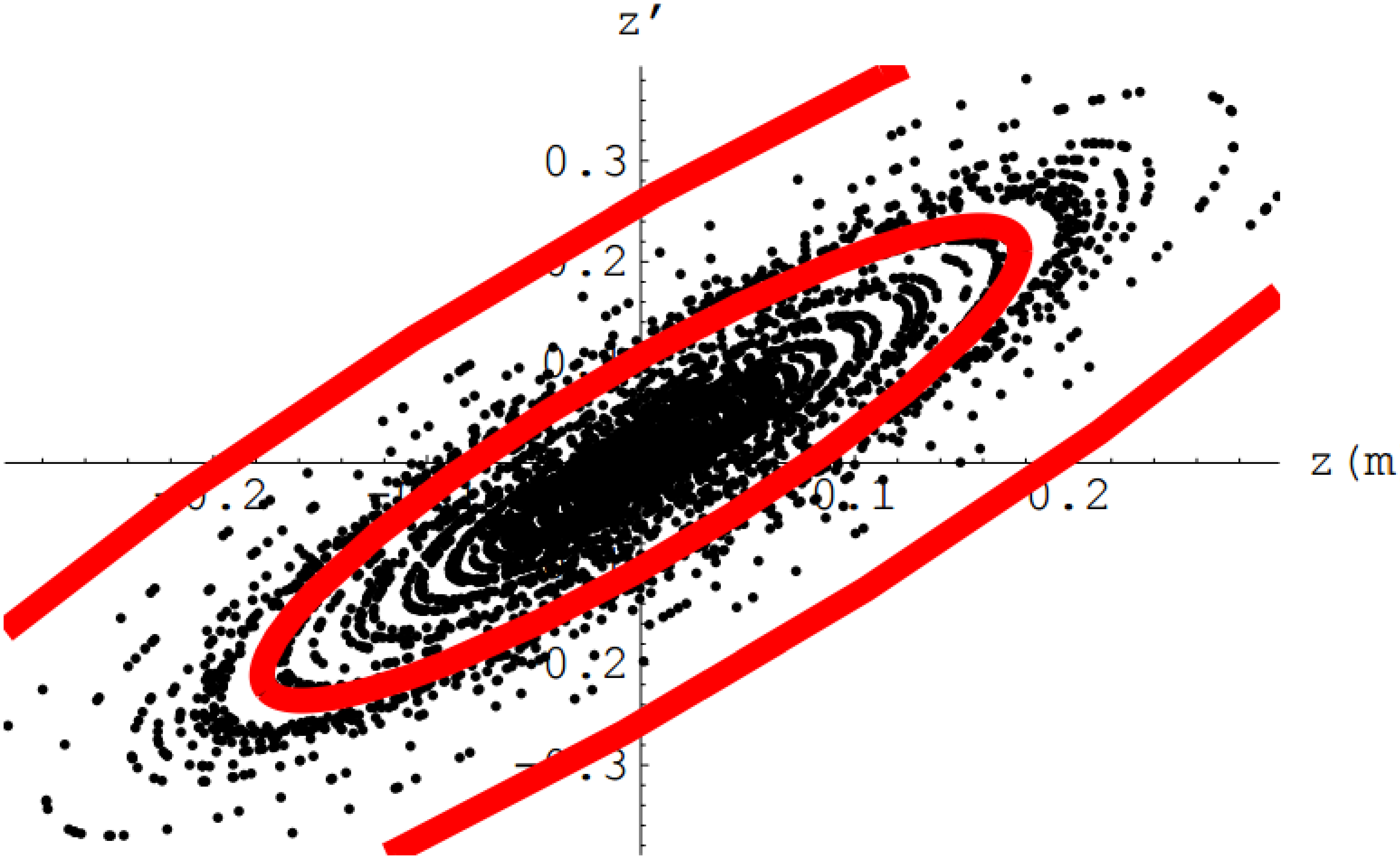
Spot size:

$$\sigma_{x,rms} = \sqrt{\beta_x\epsilon_{rms}}$$

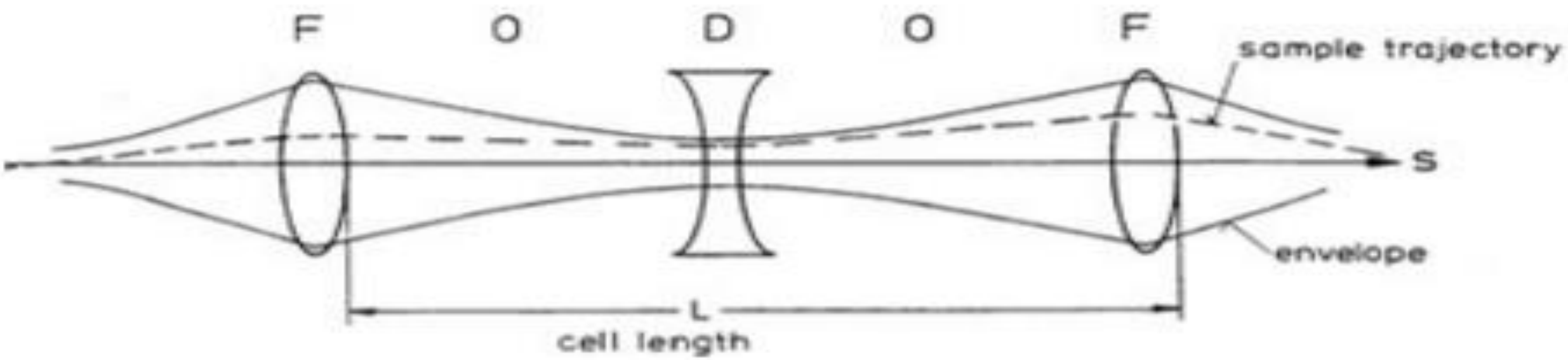
Relativistic Adiabatic Damping:

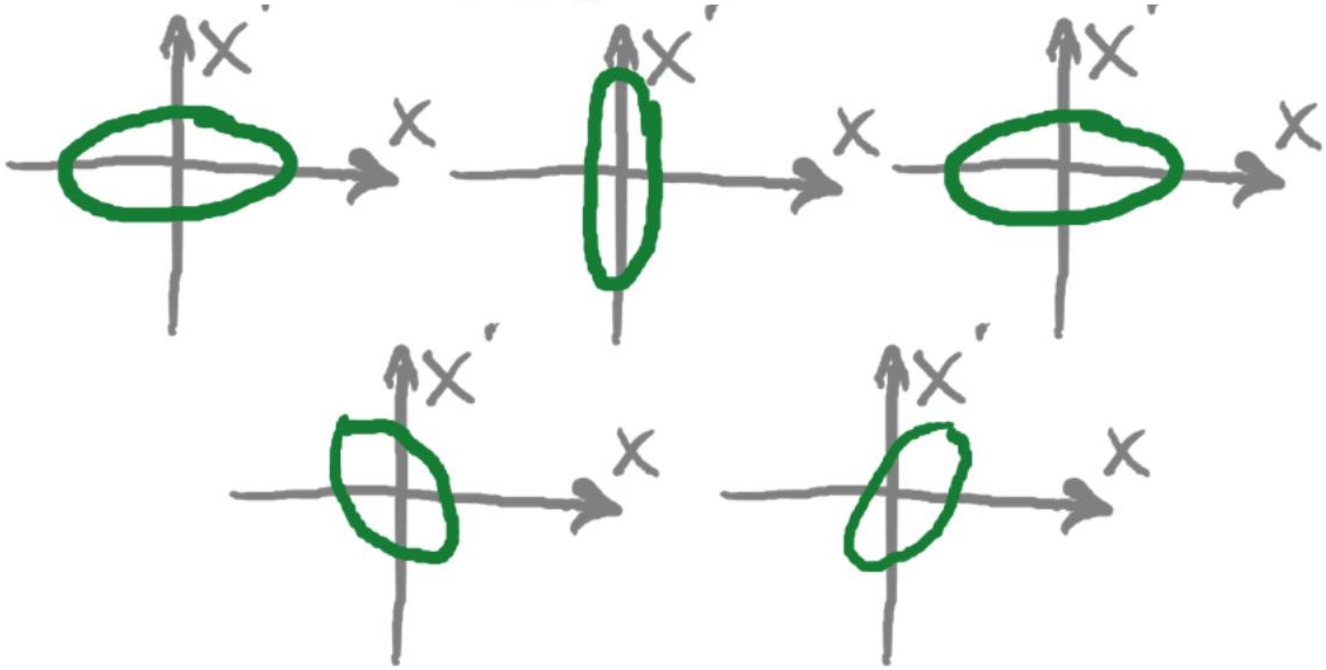
$$\epsilon_N = \beta\gamma\epsilon$$

Betatron Phase-space



Betatron Motion





Betatron Phase Advance

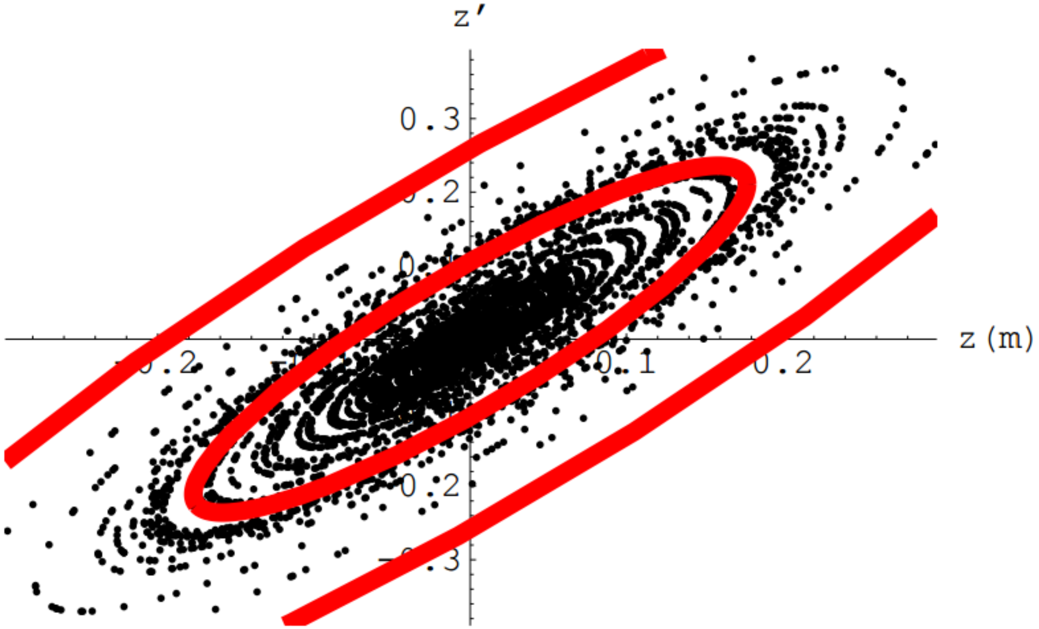
$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x) \qquad x' = -\sqrt{\frac{2J_x}{\beta_x}}[\sin(\phi_x) + \alpha_x\cos(\phi_x)]$$

$$\phi_x(s_2) - \phi_x(s_1) = \Delta\phi_x = \int_{s_1}^{s_2} \frac{1}{\beta_x(s)} ds'$$

Betatron Tune:

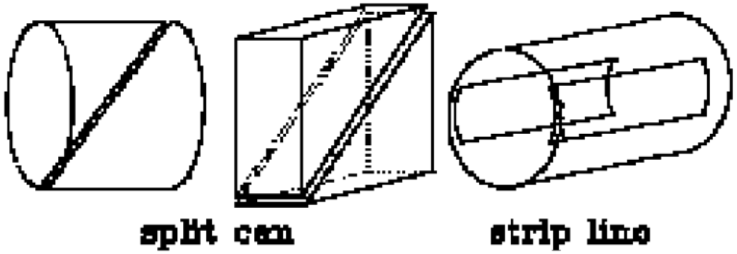
$$\nu = \frac{1}{2\pi}\int_{s_0}^{s_0+C} \frac{1}{\beta_x(s)} ds'$$

$$\phi_x(t + NT_{rev}) = \phi_x(t) + 2\pi N\nu$$

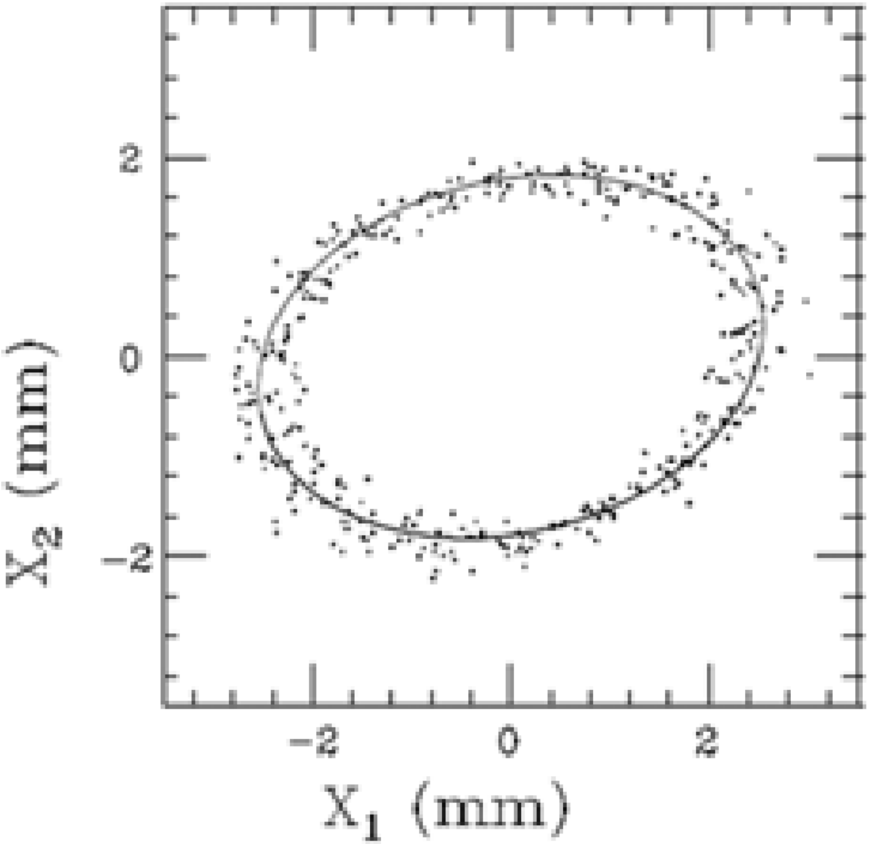


Two BPM Measurement

The correlation between X1 and X2, can be used to see the betatron phase advance between those points.



$$y \approx \frac{w}{2}\frac{U_+ - U_-}{U_+ + U_-} = \frac{w}{2}\frac{\Delta}{\Sigma},$$

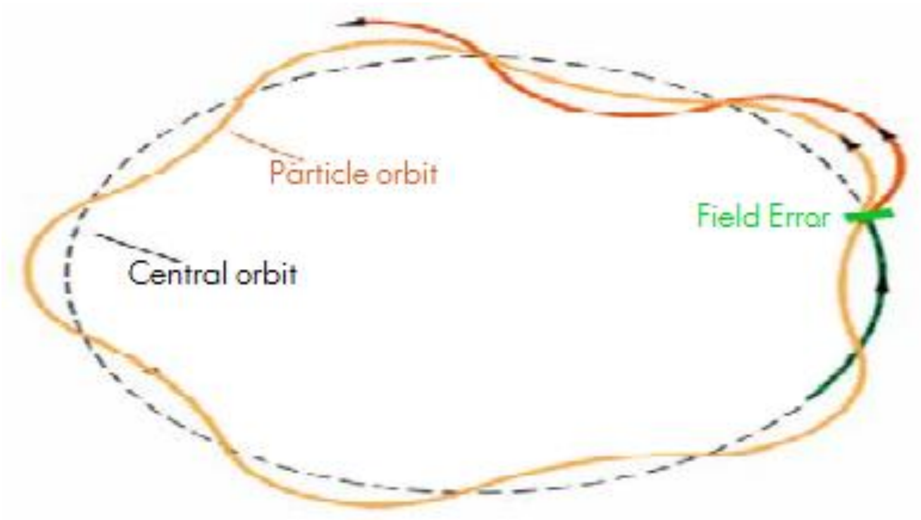


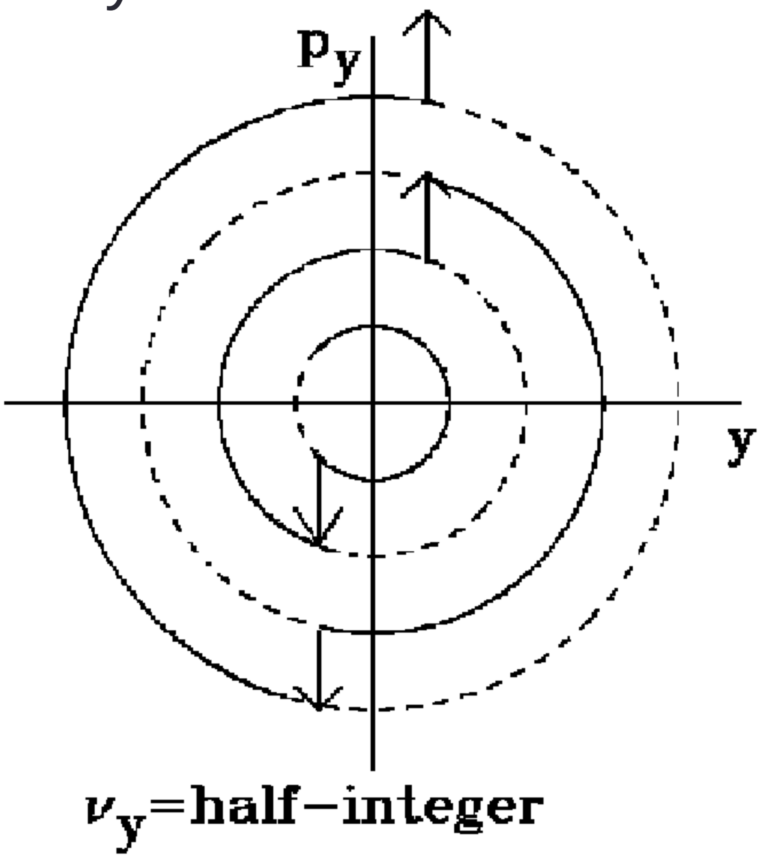
Nonlinearities

Betatron Tune Resonance

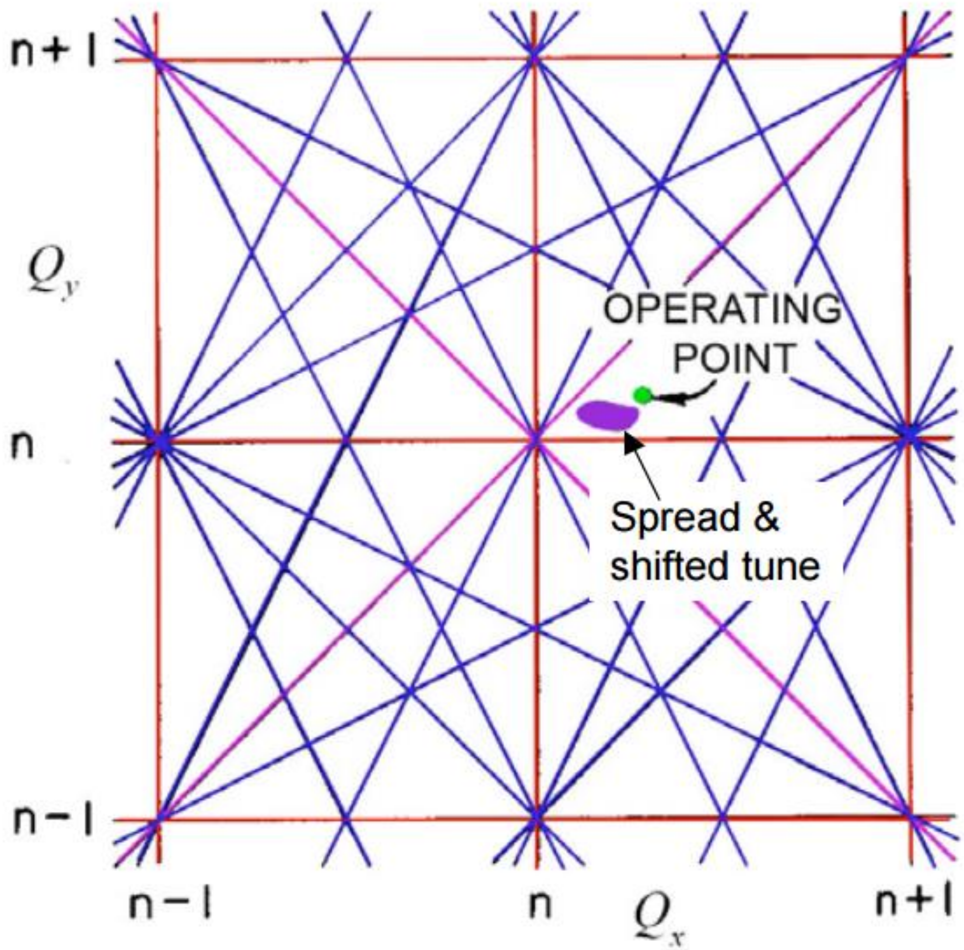
Perturbation will accumulate if tune is a fraction corresponding to the symmetry of the applied fields.

$$\nu = \frac{1}{2\pi}\int_{s_0}^{s_0+C}\frac{1}{\beta_x(s)}ds'$$





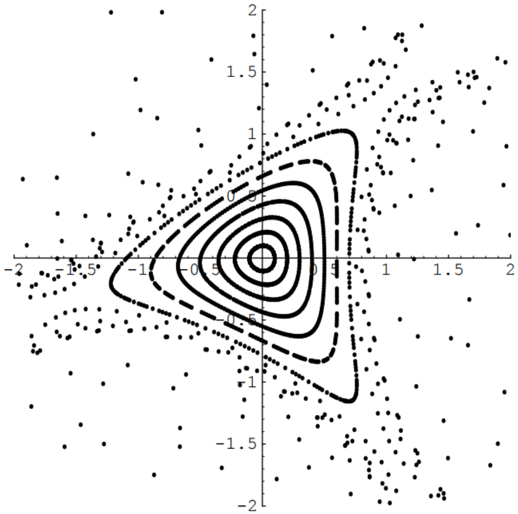
Tune Diagrams

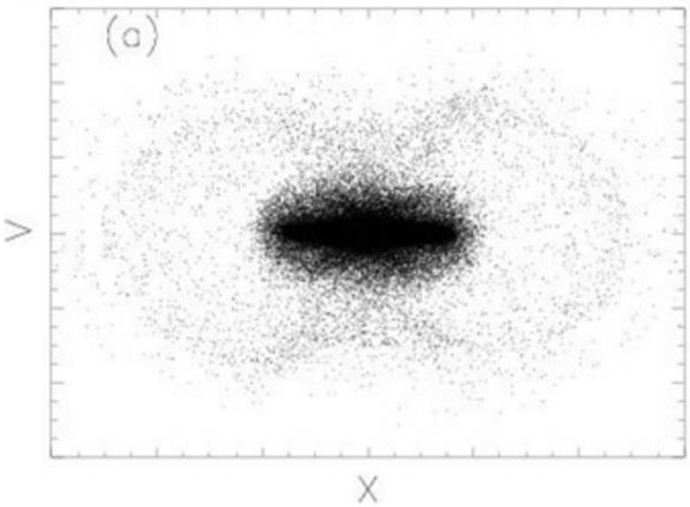


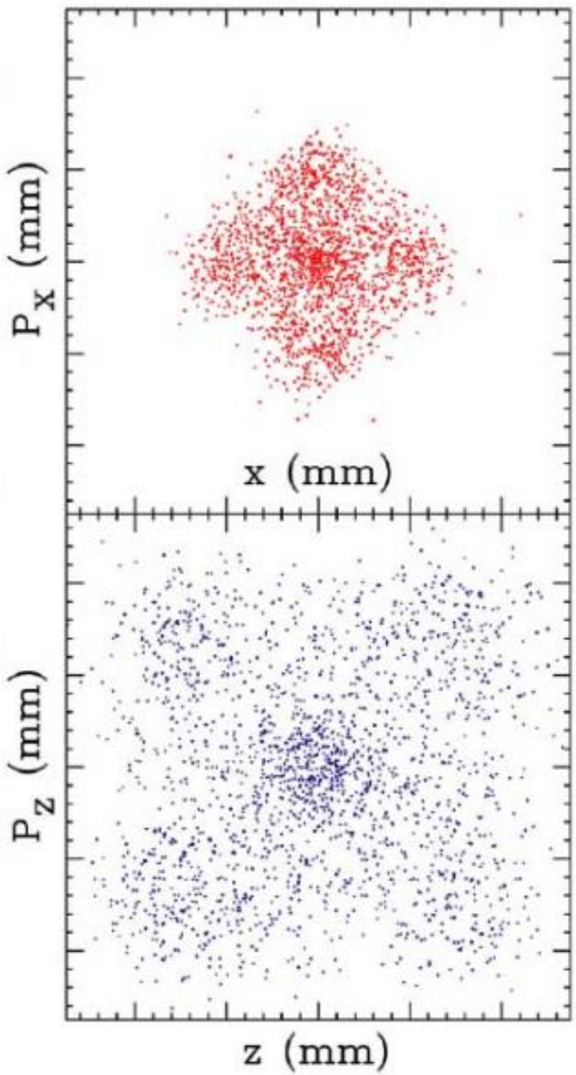
The tune is carefully picked to avoid resonances.

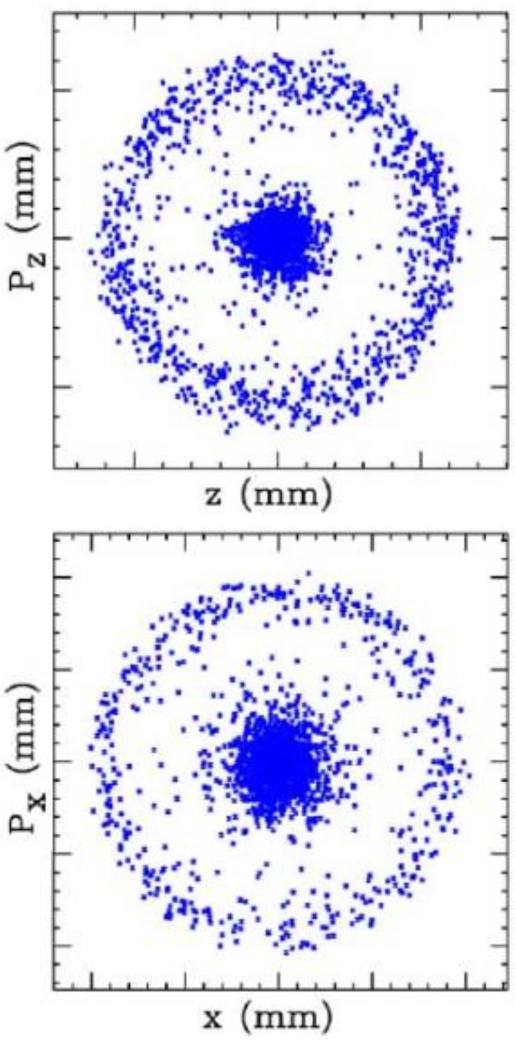
The tune for the beam occupies a finite space:
- Laslett space-charge tune spread.
- Amplitude-dependent tune shift.
- Tune-change during acceleration.
- Chromaticity
- beam-beam effects

Phase-space Distortions

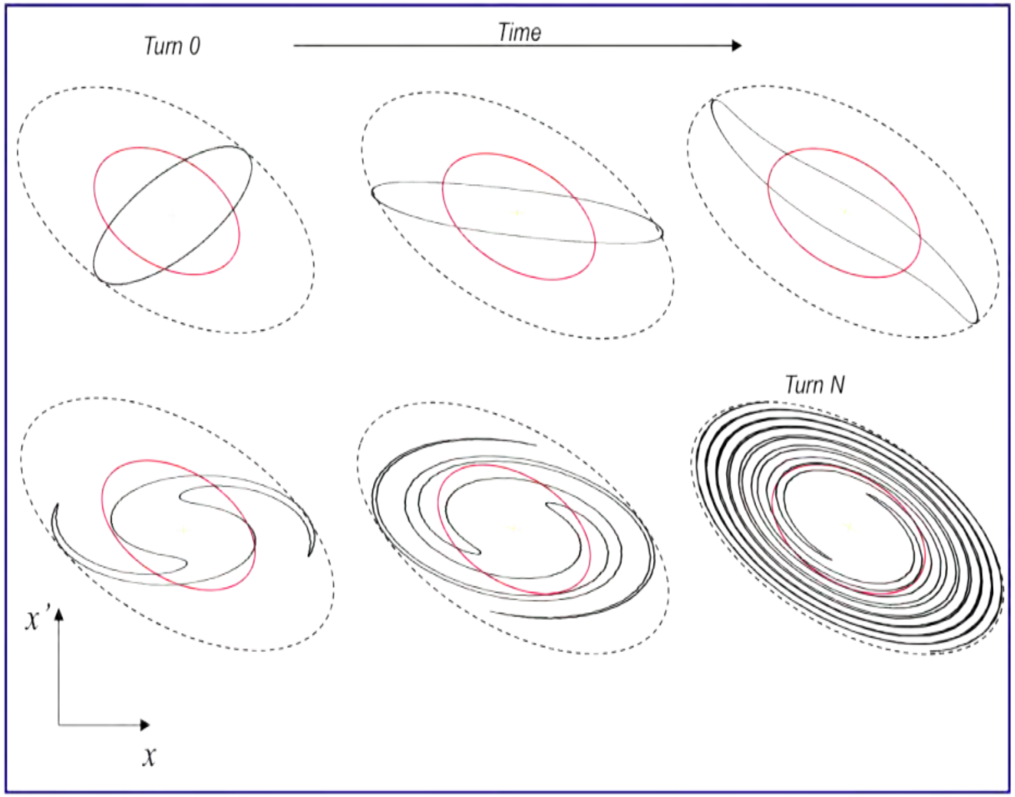








Nonlinear Decoherence

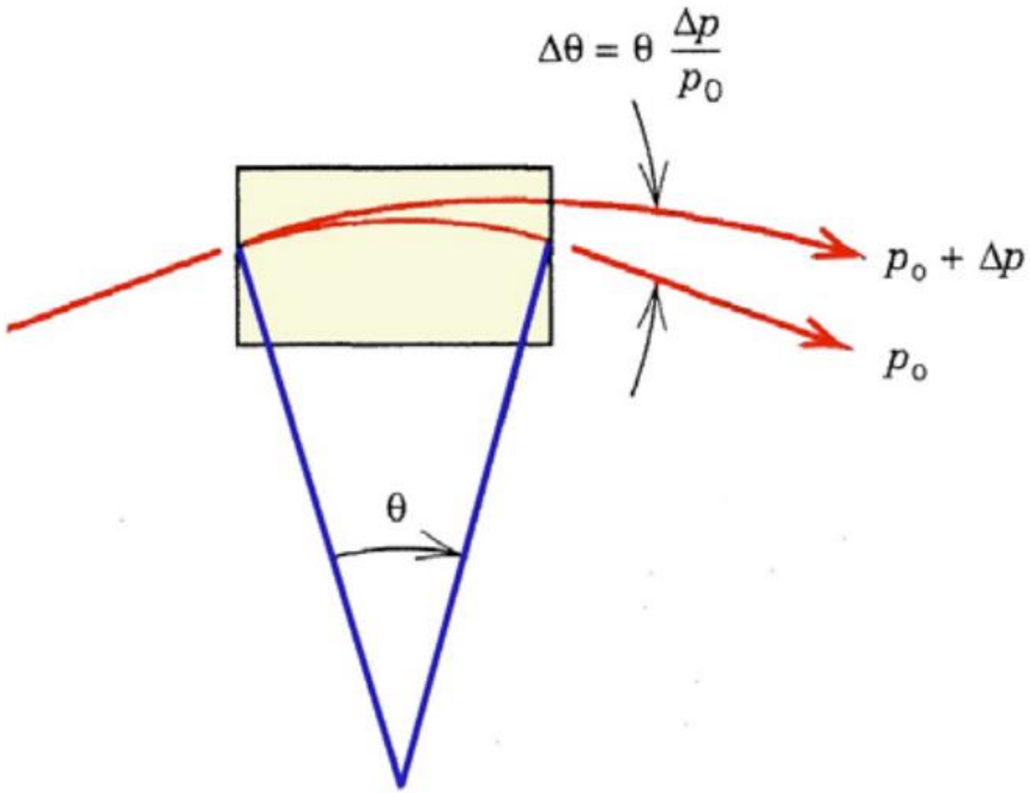


Injection errors, instabilities, and sudden lattice changes may cause a phase-space mismatch.

Tune spread causes the beam to fill-out along the phase-space contours.

Off-Momentum Particles

Dispersion



$$\delta \equiv \frac{p - p_0}{p_0}$$

Dispersion:

$$D'' + K_x(s)D = \frac{1}{\rho}$$

$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x) + D\delta$$

Spot Size:

$$\sigma^2_{x,rms} = \beta_x \epsilon_{rms} + D^2\delta^2$$

Chromaticity

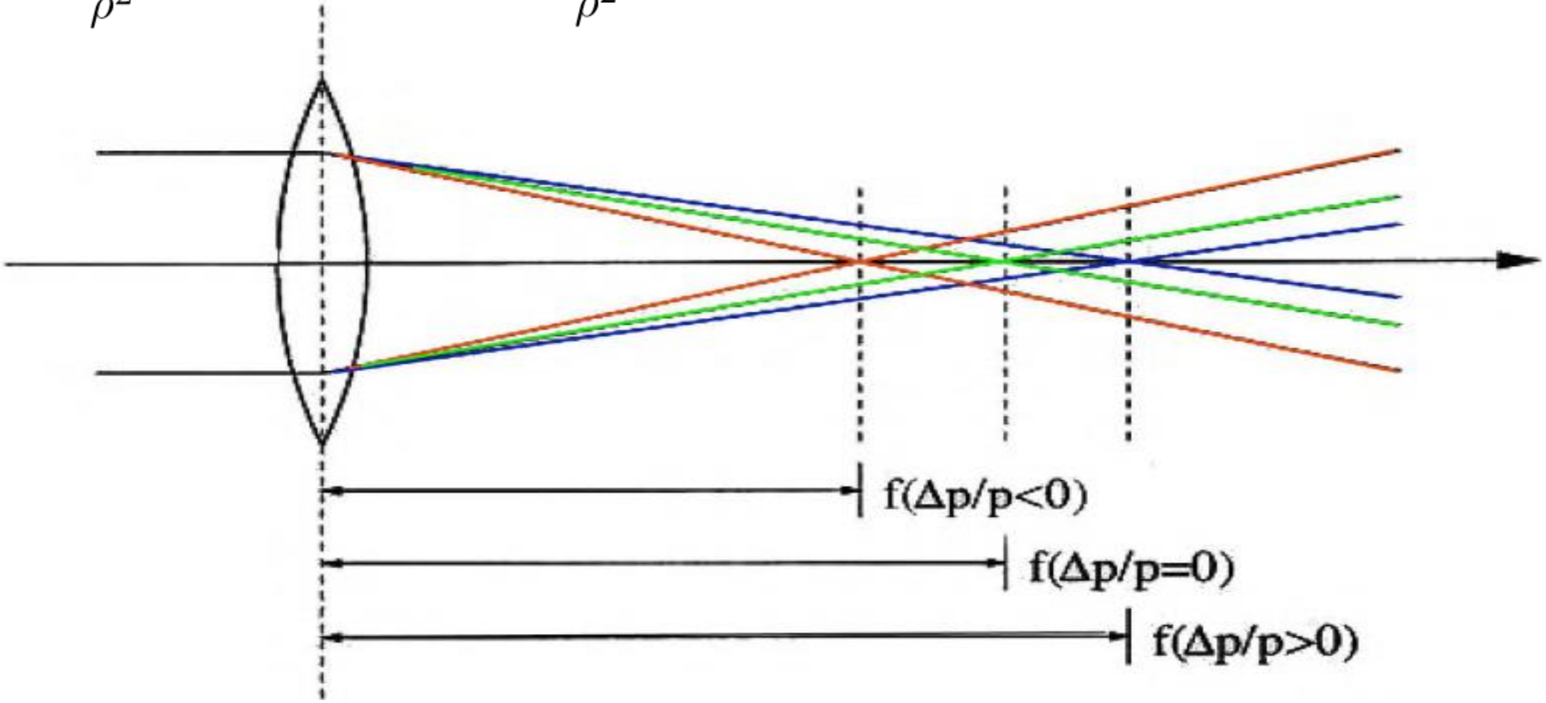
Change in tune with momentum:

$$x_\beta'' + (K_x + \Delta K_x \delta) x_\beta = 0$$

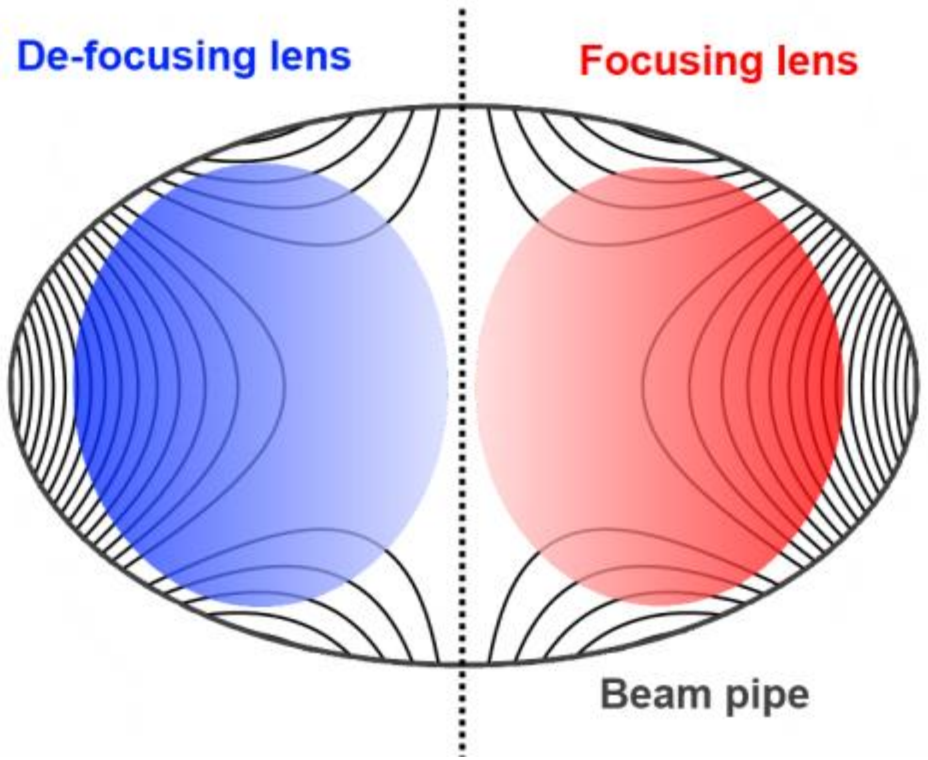
$$K_x = \frac{1}{\rho^2} - K(s) \qquad \Delta K_x = -\frac{2}{\rho^2} + K(s)$$

Chromaticity:

$$C_x = \frac{\partial}{\partial \delta} \Delta \nu_x = \frac{1}{4\pi} \int_0^C \beta_x \Delta K_x(s) ds$$



Sextupoles & Chromaticity Correction



Chromaticity is tune dependence on momentum.

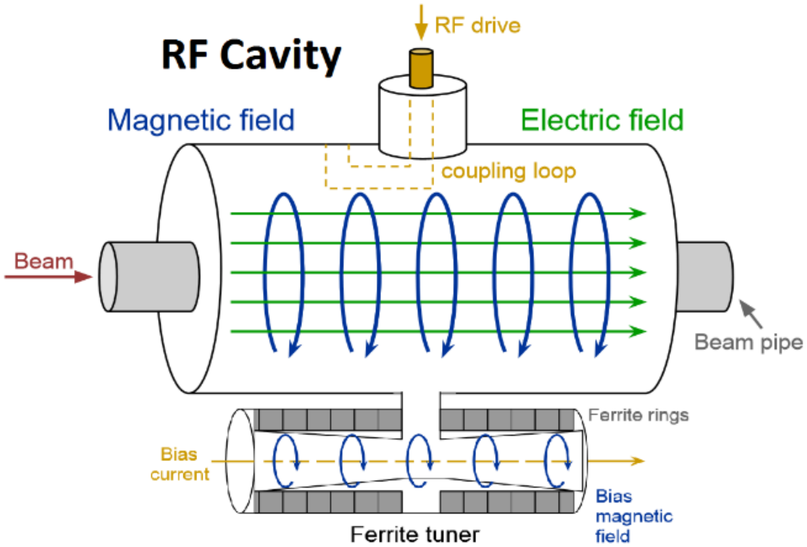
Sextupoles provide tune-shift depending on position offset.

Dispersion is position offset dependence on momentum.

$$C_x = \frac{1}{4\pi}\int_0^C \beta_x[\Delta K_x(s) + S(s)D(s)]ds$$

RF Capture

Energy in one pass through cavity



$$\Delta E = qV\frac{\beta c}{d}\int_{-d/2\beta c}^{d/2\beta c}\sin(\omega_{rf}t+\phi)dt$$

$$\Delta E = -qV\left(\frac{\beta c}{d}\right)\frac{\cos(\omega_{rf}t+\phi)}{\omega_{rf}}\Bigg|_{-d/2\beta c}^{d/2\beta c}$$

$$\chi \equiv \frac{\omega_{rf}d}{2\beta c} \qquad \Delta E = qV\left(\frac{1}{\chi}\right)\left[\frac{\cos(\phi+\chi)-\cos(\phi-\chi)}{-2}\right]$$

$$\Delta E = qV(\sin\chi/\chi)\sin(\phi) \to qV\sin(\phi)$$

Phase-Slip Factor **η**

The arrival time of the particle depends on the momentum:

$$\delta \equiv \frac{p - p_0}{p_0} \qquad \frac{T - T_{rev}}{T_{rev}} \approx 0 + \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} \delta = \eta \delta$$

Higher momentum particles may arrive earlier or later than lower momentum particles:

$$\eta = \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} = \frac{1}{C} \frac{\partial C}{\partial \delta} - \frac{1}{\beta} \frac{\partial \beta}{\partial \delta} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}$$

We can write the change in phase per unit time using the phase-slip factor:

$$\dot{\phi} = f_{rev} \Delta \phi = 2\pi f_{rev} \frac{\Delta T}{T_{rf}} = 2\pi f_{rev} h \frac{\Delta T}{T_{rev}} = 2\pi f_{rev} h \eta \delta \qquad f_{rf} = h f_{rev}$$

Longitudinal Focusing

$$\dot{\phi} = 2\pi f_{rev} h\eta\delta, \;\; \dot{\delta} = f_{rev}\frac{qV}{\beta^2 E_0}\sin(\phi)$$

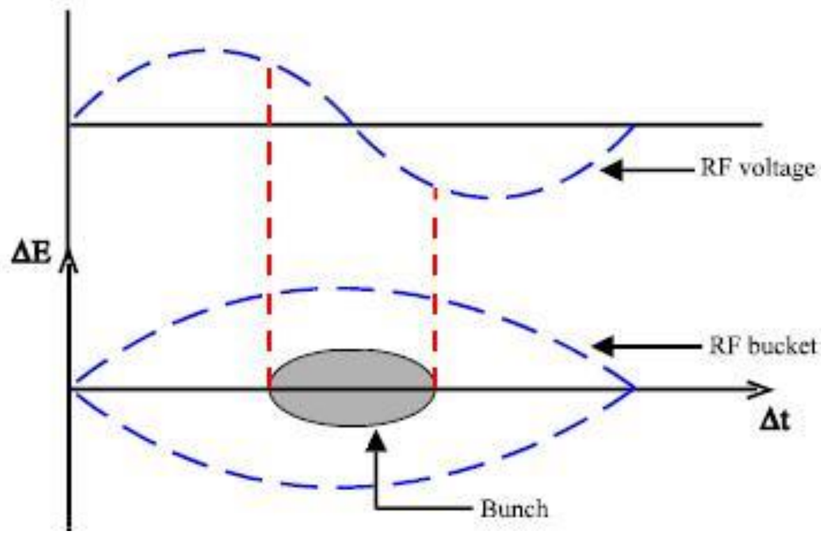
$$\ddot{\phi} = 2\pi f_{rev}^2 \frac{qV}{\beta^2 E_0} h\eta \sin(\phi)$$

Synchrotron Freq.

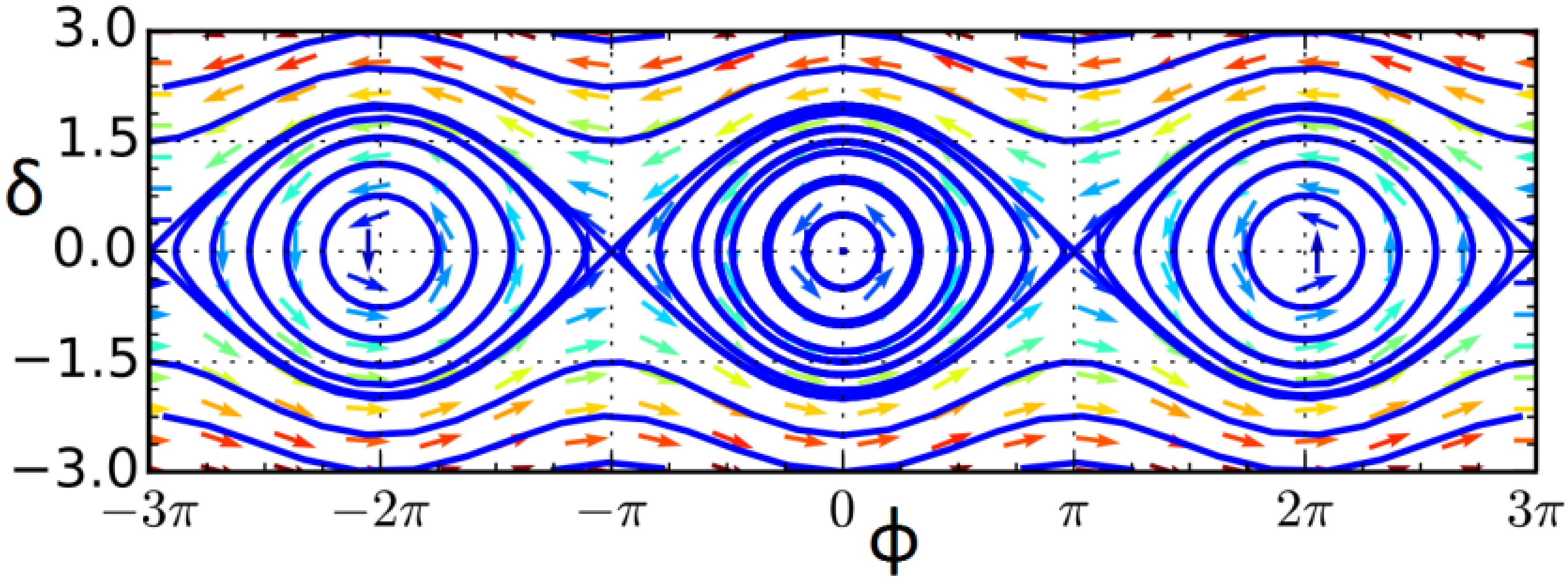
$$\eta < 0 : \ddot{\phi} = -\omega_s^2 \sin(\phi)$$

$$\eta > 0 : \ddot{\phi} = -\omega_s^2 \sin(\phi + \pi)$$

$$\omega_s = 2\pi f_{rev}\sqrt{\frac{qVh|\eta|}{2\pi\beta^2 E_0}}$$



Oscillatory & Slipping Motion

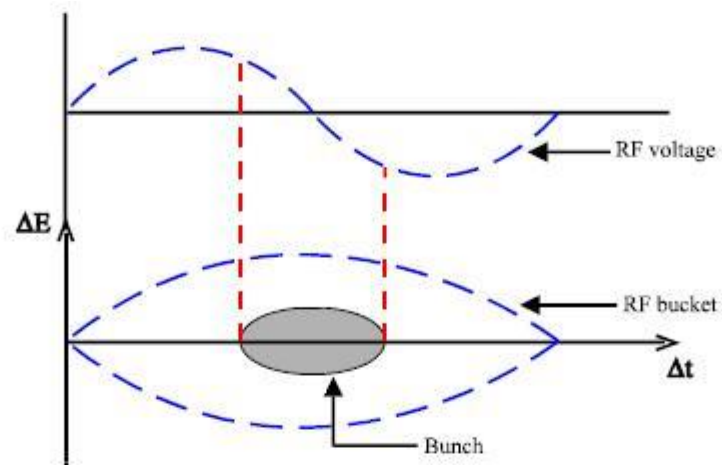


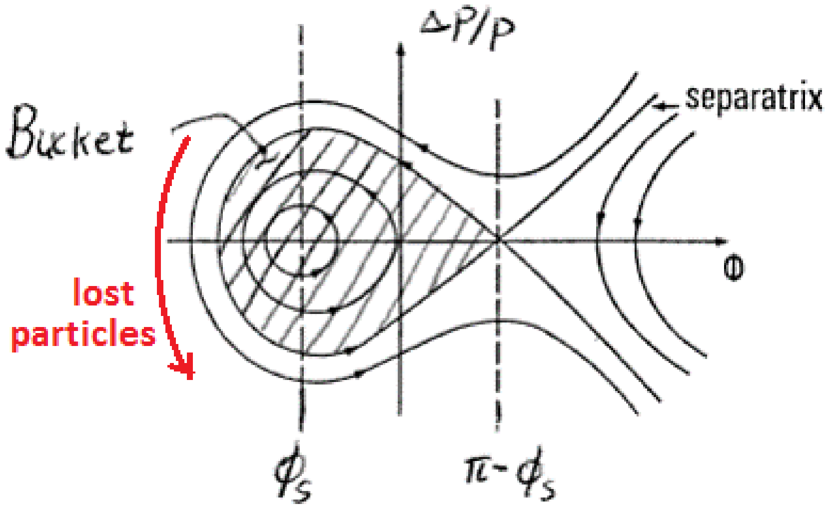
The further off momentum, the faster the slipping motion.

Lost particles rapidly decohere from each other.

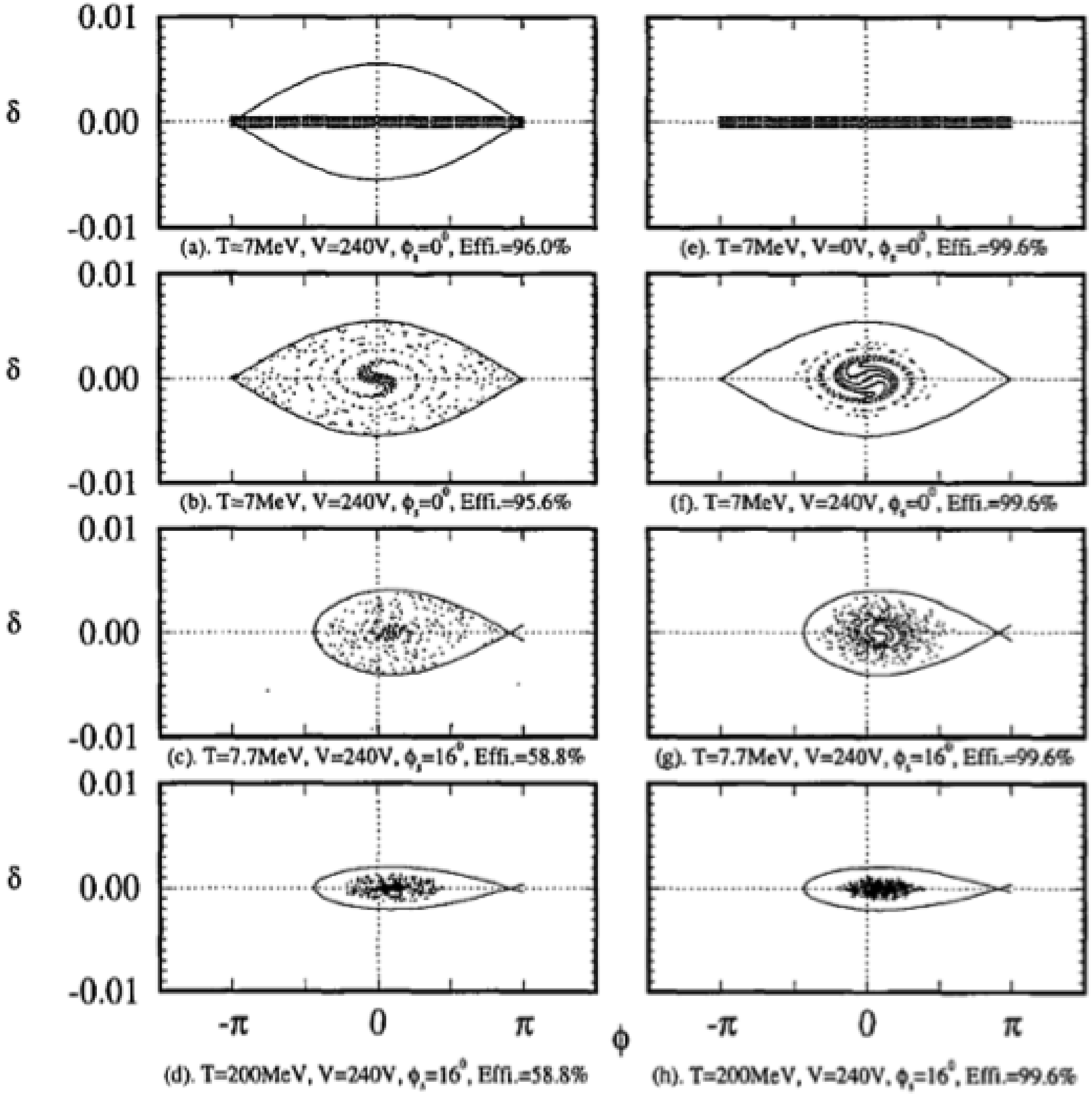
RF Acceleration

- Particles in the bucket can be accelerated by adiabatically changing the RF frequency.
- Particles outside the bucket are left behind, decelerated relative to the moving reference frame.





$$\dot{\delta} = f_{rev} V_{\delta} [\sin(\phi) - \sin(\phi_s)], \;\; \dot{\phi} = 2\pi f_{rev} h \eta \delta$$

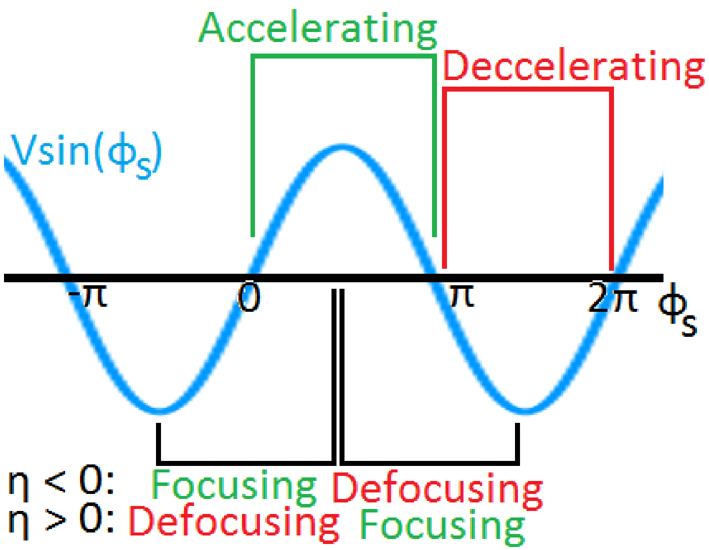


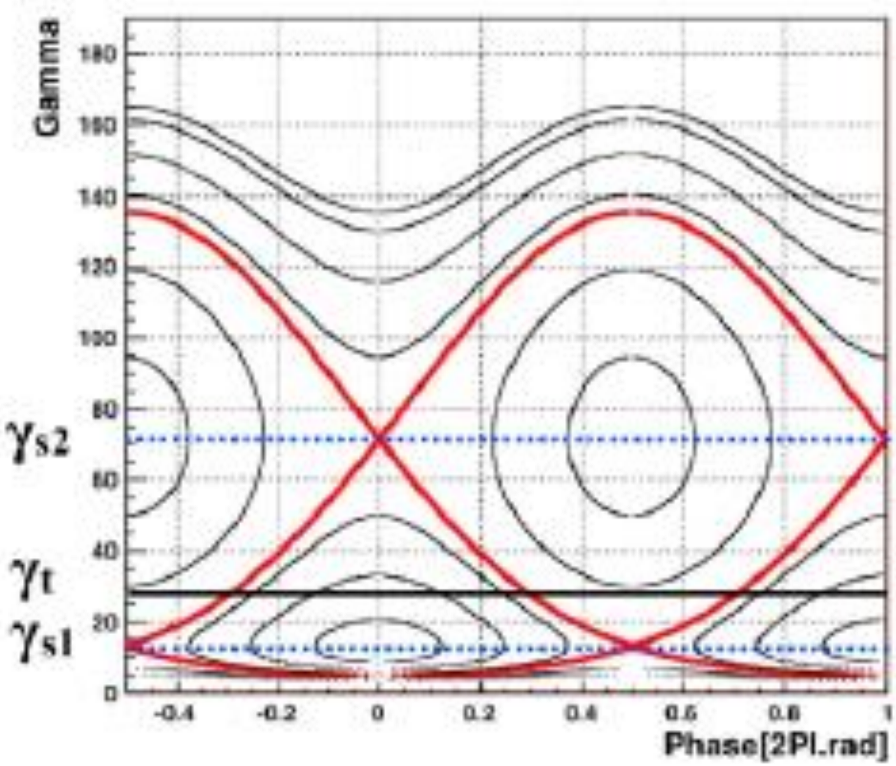
Non-Adiabactic

Adiabactic

X. Kang
SY. Lee

Transition Crossing





$$\frac{T - T_{rev}}{T_{rev}} \approx 0 + \frac{1}{T_{rev}}\frac{\partial T}{\partial \delta}\delta + \frac{1}{T_{rev}}\frac{\partial^2 T}{\partial \delta^2}\frac{\delta^2}{2} = \eta_0\delta + \eta_1\delta^2$$